Eletromagnetismo

Johny Carvalho Silva*
Universidade Federal do Rio Grande
Instituto de Matemática, Física e Estatística

Lista - 1

1.1 - Mostrar que

$$
\begin{aligned}
(\mathbf{A}\times\mathbf{B})_i &= \epsilon_{ijk}A_jB_k &(1)\\
(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} &= \epsilon_{ijk}A_iB_jC_k &(2)
\end{aligned}
$$

....:: Solução (1) ::..

$$
\begin{aligned}
(\mathbf{A}\times\mathbf{B}) &= \hat{\mathbf{e}}_iA_i\times\hat{\mathbf{e}}_jB_j = A_iB_j\,\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j = \epsilon_{ijk}A_iB_j\,\hat{\mathbf{e}}_k\\
(\mathbf{A}\times\mathbf{B})\cdot\hat{\mathbf{e}}_k &= \epsilon_{ijk}A_iB_j\,\hat{\mathbf{e}}_k\cdot\hat{\mathbf{e}}_k\\
(\mathbf{A}\times\mathbf{B})_k &= \epsilon_{ijk}A_iB_j \quad \text{como os índices são mudos posso fazer}\\
(\mathbf{A}\times\mathbf{B})_i &= \epsilon_{ijk}A_jB_k
\end{aligned}
$$

....:: Solução (2) ::..

$$
\begin{aligned}
(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} &= (\hat{\mathbf{e}}_iA_i\times\hat{\mathbf{e}}_jB_j)\cdot\hat{\mathbf{e}}_k\,C_k = (A_iB_j\,\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j)\cdot\hat{\mathbf{e}}_k\,C_k = A_iB_j\,\epsilon_{ijl}\,\hat{\mathbf{e}}_l\cdot\hat{\mathbf{e}}_k\,C_k\\
&= A_iB_jC_k\,\epsilon_{ijl}\,\delta_{lk} = A_iB_jC_k\,\epsilon_{ijk}\\
&= \epsilon_{ijk}\,A_iB_j\,C_k
\end{aligned}
$$

1.2 - Mostrar que

$$(\mathbf{A}\cdot\nabla)\mathbf{A} = -\mathbf{A}\times(\nabla\times\mathbf{A}) \quad \text{quando} \quad \mathbf{A}^2 = const \tag{3}$$

....:: Solução (3) ::..

$$
\begin{aligned}
-\mathbf{A}\times(\nabla\times\mathbf{A}) &= -\hat{\mathbf{e}}_iA_i\times(\hat{\mathbf{e}}_j\partial_j\times\hat{\mathbf{e}}_kA_k) = -\hat{\mathbf{e}}_iA_i\times(\partial_jA_k\,\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_k)\\
&= -\hat{\mathbf{e}}_iA_i\times(\partial_jA_k\,\epsilon_{jkl}\hat{\mathbf{e}}_l) = -A_i\partial_jA_k\,\epsilon_{jkl}\,\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_l\\
&= -A_i\partial_jA_k\,\epsilon_{jkl}\,\epsilon_{ilm}\,\hat{\mathbf{e}}_m = \epsilon_{jkl}\,\epsilon_{iml}\,A_i\partial_jA_k\,\hat{\mathbf{e}}_m\\
&= A_i\partial_jA_k\,\hat{\mathbf{e}}_m(\delta_{ji}\delta_{km}-\delta_{jm}\delta_{ki}) = A_i\partial_jA_k\,\hat{\mathbf{e}}_m\delta_{ji}\delta_{km} - A_i\partial_jA_k\delta_{jm}\delta_{ki}\,\hat{\mathbf{e}}_m\\
&= A_j\partial_jA_m\,\hat{\mathbf{e}}_m - A_k\partial_mA_k\,\hat{\mathbf{e}}_m = (\mathbf{A}\cdot\nabla)\mathbf{A} - \nabla\mathbf{A}^2\\
&= (\mathbf{A}\cdot\nabla)\mathbf{A}
\end{aligned}
$$

1.3 - Mostrar que se o tensor S_{ik} é simétrico e o tensor A_{ik} é antisimétrico, temos

$$S_{ik}A_{ik} = 0 \tag{4}$$

*johny@johnycarvalho.com

....:: Solução (4) ::...

$$Q \equiv S_{ik}A_{ik} = -S_{ki}A_{ik} = -Q \quad \Longrightarrow \quad 2Q = 0 \quad \Longrightarrow \quad Q = 0$$

1.4 - Demonstrar as identidades abaixo

$$\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi \tag{5}$$
$$\nabla\cdot(\varphi\mathbf{A}) = \varphi(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla\varphi) \tag{6}$$
$$\nabla\times(\varphi\mathbf{A}) = \varphi(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla\varphi) \tag{7}$$
$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} \tag{8}$$
$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \tag{9}$$
$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{10}$$
$$(\nabla\cdot\mathbf{A})\mathbf{B} = (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}(\nabla\cdot\mathbf{A}) \tag{11}$$
$$(\mathbf{A}\times\nabla)\times\mathbf{B} = (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}\times(\nabla\times\mathbf{B}) - \mathbf{A}\cdot(\nabla\cdot\mathbf{B}) \tag{12}$$

....:: Solução (5) ::...

$$\begin{aligned}\nabla(\varphi\psi) &= \hat{\mathbf{e}}_i\partial_i\varphi_i\psi_i = \hat{\mathbf{e}}_i(\varphi_i\partial_i\psi_i + \psi_i\partial_i\varphi_i) = \varphi_i\hat{\mathbf{e}}_i\partial_i\psi_i + \psi_i\hat{\mathbf{e}}_i\partial_i\varphi_i \\ &= \varphi\nabla\psi + \psi\nabla\varphi \end{aligned}\tag{13}$$

....:: Solução (6) ::...

$$\begin{aligned}\nabla\cdot(\varphi\mathbf{A}) &= \hat{\mathbf{e}}_i\partial_i\cdot(\varphi_i\hat{\mathbf{e}}_jA_j) = \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\partial_i\varphi_iA_j = \delta_{ij}\partial_i\varphi_iA_j = \partial_i\varphi_iA_i = \varphi_i\partial_iA_i + A_i\partial_i\varphi_i \\ &= \varphi(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla\varphi)\end{aligned}\tag{14}$$

....:: Solução (7) ::...

$$\begin{aligned}\nabla\times(\varphi\mathbf{A}) &= \hat{\mathbf{e}}_i\partial_i\times(\varphi_i\hat{\mathbf{e}}_jA_j) = \hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j\partial_i\varphi_iA_j = \epsilon_{ijk}\,\hat{\mathbf{e}}_k\,\partial_i\varphi_iA_j = \epsilon_{ijk}\,\hat{\mathbf{e}}_k\,\partial_i\varphi_iA_j \\ &= \epsilon_{ijk}\,\hat{\mathbf{e}}_k(\varphi_i\partial_iA_j + A_j\partial_i\varphi_i) = \epsilon_{ijk}\varphi_i\partial_iA_j\,\hat{\mathbf{e}}_k - \epsilon_{jik}\,A_j\partial_i\varphi_i\hat{\mathbf{e}}_k \\ &= \varphi(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla\varphi)\end{aligned}\tag{15}$$

....:: Solução (8) ::...
nas identidades acima o operador nabla executou duas operações: operação diferencial e operação vetorial. Para demonstrar a eq.(8) farei a seguinte definição,

$$def\begin{cases}\nabla(\mathbf{A}\cdot\check{\mathbf{B}}) \equiv \text{o símbolo } (\,\check{}\,) \text{ indica que o operador nabla atua somente no vetor } \mathbf{B} \\ \nabla(\check{\mathbf{A}}\cdot\mathbf{B}) \equiv \text{o símbolo } (\,\check{}\,) \text{ indica que o operador nabla atua somente no vetor } \mathbf{A}\end{cases}\tag{16}$$

operação diferencial,

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \nabla(\mathbf{A}\cdot\check{\mathbf{B}}) + \nabla(\check{\mathbf{A}}\cdot\mathbf{B}) \tag{17}$$

o primeiro termo da eq. (17) pode ser encontrado fazendo $\mathbf{a}\equiv\mathbf{A}$, $\mathbf{b}\equiv\mathbf{B}$ e $\mathbf{C}\equiv\nabla$ e usando a definição (16) no produto triplo abaixo,

operação vetorial,

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= -\mathbf{c}(\mathbf{a}\cdot\mathbf{b}) + \mathbf{b}(\mathbf{c}\cdot\mathbf{a}) \\ \mathbf{A}\times(\check{\mathbf{B}}\times\nabla) &= -\nabla(\mathbf{A}\cdot\check{\mathbf{B}}) + \check{\mathbf{B}}(\nabla\cdot\mathbf{A}) \\ \nabla(\mathbf{A}\cdot\check{\mathbf{B}}) &= \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{A}\cdot\nabla)\mathbf{B}\end{aligned}\tag{18}$$

o segundo termo da eq. (17) pode ser encontrado procedendo de modo análogo,

$$\begin{aligned} \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= -\mathbf{c}(\mathbf{a}\cdot\mathbf{b})+\mathbf{b}(\mathbf{c}\cdot\mathbf{a}) \\ \mathbf{B}\times(\nabla\times\check{\mathbf{A}}) &= -\check{\mathbf{A}}(\mathbf{B}\cdot\nabla)+\nabla(\check{\mathbf{A}}\cdot\mathbf{B}) \\ \nabla(\check{\mathbf{A}}\cdot\mathbf{B}) &= \mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A} \end{aligned} \tag{19}$$

das eqs. (17), (18) e (19),

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A} \tag{20}$$

....:: Solução (10) ::..

operação diferencial,

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \nabla\times(\mathbf{A}\times\check{\mathbf{B}})+\nabla\times(\check{\mathbf{A}}\times\mathbf{B}) \tag{21}$$

operação vetorial,

$$\begin{aligned} \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= -\mathbf{c}(\mathbf{a}\cdot\mathbf{b})+\mathbf{b}(\mathbf{c}\cdot\mathbf{a}) \\ \nabla\times(\mathbf{A}\times\check{\mathbf{B}}) &= -\check{\mathbf{B}}(\nabla\cdot\mathbf{A})+\mathbf{A}(\nabla\cdot\check{\mathbf{B}}) \\ &= \mathbf{A}(\nabla\cdot\mathbf{B})-(\mathbf{A}\cdot\nabla)\mathbf{B} \end{aligned} \tag{22}$$

$$\begin{aligned} \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= -\mathbf{c}(\mathbf{a}\cdot\mathbf{b})+\mathbf{b}(\mathbf{c}\cdot\mathbf{a}) \\ \nabla\times(\check{\mathbf{A}}\times\mathbf{B}) &= -\mathbf{B}(\nabla\cdot\check{\mathbf{A}})+\check{\mathbf{A}}(\mathbf{B}\cdot\nabla) \\ &= (\mathbf{B}\cdot\nabla)\mathbf{A}-\mathbf{B}(\nabla\cdot\mathbf{A}) \end{aligned} \tag{23}$$

das eqs. (21), (22) e (23),

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B} \tag{24}$$

....:: Solução (11) ::..

operação diferencial,

$$(\nabla\cdot\mathbf{A})\mathbf{B} = (\nabla\cdot\mathbf{A})\check{\mathbf{B}}+(\nabla\cdot\check{\mathbf{A}})\mathbf{B} \tag{25}$$

operação vetorial,

$$(\nabla\cdot\mathbf{A})\check{\mathbf{B}} = (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{26}$$

$$(\nabla\cdot\check{\mathbf{A}})\mathbf{B} = \mathbf{B}(\nabla\cdot\mathbf{A}) \tag{27}$$

das eqs. (25), (26) e (27),

$$(\nabla\cdot\mathbf{A})\mathbf{B} = (\mathbf{A}\cdot\nabla)\mathbf{B}+\mathbf{B}(\nabla\cdot\mathbf{A}) \tag{28}$$

....:: Solução (12) ::..

operação diferencial,

$$(\mathbf{A}\times\nabla)\times\mathbf{B} = (\mathbf{A}\times\nabla)\times\check{\mathbf{B}} \tag{29}$$

operação vetorial,

$$\begin{aligned} \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= -\mathbf{c}(\mathbf{a}\cdot\mathbf{b})+\mathbf{b}(\mathbf{c}\cdot\mathbf{a}) \\ (\mathbf{A}\times\nabla)\times\check{\mathbf{B}} = -\check{\mathbf{B}}\times(\mathbf{A}\times\nabla) &= +\nabla(\check{\mathbf{B}}\cdot\mathbf{A})-\mathbf{A}(\nabla\cdot\check{\mathbf{B}}) \end{aligned} \tag{30}$$

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= -\mathbf{c}(\mathbf{a}\cdot\mathbf{b})+\mathbf{b}(\mathbf{c}\cdot\mathbf{a})\\ \mathbf{A}\times(\nabla\times\mathbf{B}) &= -\mathbf{B}(\mathbf{A}\cdot\nabla)+\nabla(\check{\mathbf{B}}\cdot\mathbf{A})\\ \mathbf{A}\times(\nabla\times\mathbf{B})+(\mathbf{A}\cdot\nabla)\mathbf{B} &= \nabla(\check{\mathbf{B}}\cdot\mathbf{A}) \qquad (31)\end{aligned}$$

das eqs. (29), (30) e (31),

$$(\mathbf{A}\times\nabla)\times\mathbf{B} = \mathbf{A}\times(\nabla\times\mathbf{B})+(\mathbf{A}\cdot\nabla)\mathbf{B}-\mathbf{A}\cdot(\nabla\cdot\mathbf{B}) \qquad (32)$$

1.5 - Calcule as equações abaixo usando as coordenadas cartesianas, cilindricas e esféricas

$$\nabla\cdot\mathbf{r} \qquad (33)$$
$$\nabla\times\mathbf{r} \qquad (34)$$
$$\nabla(\mathbf{1}\cdot\mathbf{r}) \qquad (35)$$
$$(\mathbf{1}\cdot\nabla)\mathbf{r} \qquad (36)$$

onde $\mathbf{r}$ é o raio vetor, $\mathbf{1}$ é vetor constate.

....:: Solução (33) ::...

coordenadas cartesianas

$$\nabla\cdot\mathbf{r} = \hat{\mathbf{e}}_i\partial_i\cdot\hat{\mathbf{e}}_j r_j = \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\partial_i r_j = \delta_{ij}\partial_i r_j = \partial_i r_i = 3 \qquad (37)$$

coordenadas cilíndricas

$$\begin{aligned}\nabla\cdot\mathbf{r} &= \left(\hat{\mathbf{e}}_\varrho\partial_\varrho+\frac{1}{\varrho}\hat{\mathbf{e}}_\theta\partial_\theta+\hat{\mathbf{e}}_z\partial_z\right)\cdot\left(\varrho\hat{\mathbf{e}}_\varrho+z\hat{\mathbf{e}}_z\right)\\ &= \hat{\mathbf{e}}_\varrho\partial_\varrho\cdot\varrho\hat{\mathbf{e}}_\varrho+\frac{1}{\varrho}\hat{\mathbf{e}}_\theta\partial_\theta\cdot\varrho\hat{\mathbf{e}}_\varrho+\hat{\mathbf{e}}_z\partial_z\cdot\varrho\hat{\mathbf{e}}_\varrho+\hat{\mathbf{e}}_\varrho\partial_\varrho\cdot z\hat{\mathbf{e}}_z+\frac{1}{\varrho}\hat{\mathbf{e}}_\theta\partial_\theta\cdot z\hat{\mathbf{e}}_z+\hat{\mathbf{e}}_z\partial_z\cdot z\hat{\mathbf{e}}_z\\ &= \hat{\mathbf{e}}_\varrho\cdot\hat{\mathbf{e}}_\varrho+\frac{1}{\varrho}\varrho\hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{e}}_\theta+\hat{\mathbf{e}}_z\cdot\hat{\mathbf{e}}_z\\ &= 1+1+1=3\end{aligned}$$

coordenadas esféricas

$$\begin{aligned}\nabla\cdot\mathbf{r} &= \left(\hat{\mathbf{e}}_r\partial_r+\frac{1}{r}\hat{\mathbf{e}}_\theta\partial_\theta+\frac{1}{r\sin\theta}\hat{\mathbf{e}}_\phi\partial_\phi\right)\cdot\left(r\hat{\mathbf{e}}_r\right)\\ &= \hat{\mathbf{e}}_r\partial_r\cdot r\hat{\mathbf{e}}_r+\frac{1}{r}\hat{\mathbf{e}}_\theta\partial_\theta\cdot r\hat{\mathbf{e}}_r+\frac{1}{r\sin\theta}\hat{\mathbf{e}}_\phi\partial_\phi\cdot r\hat{\mathbf{e}}_r\\ &= \hat{\mathbf{e}}_r\cdot\hat{\mathbf{e}}_r+\frac{1}{r}r\hat{\mathbf{e}}_\theta\cdot\hat{\mathbf{e}}_\theta+\frac{1}{r\sin\theta}r\sin\theta\hat{\mathbf{e}}_\phi\cdot\hat{\mathbf{e}}_\phi\\ &= 1+1+1=3 \qquad (38)\end{aligned}$$

....:: Solução (34) ::...

$$\nabla\times\mathbf{r} = \hat{\mathbf{e}}_i\partial_i\times\hat{\mathbf{e}}_j r_j = \hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_j\partial_i r_j = \epsilon_{ijk}\hat{\mathbf{e}}_k\partial_i r_j = \epsilon_{ijk}\hat{\mathbf{e}}_k\delta_{ij} = \epsilon_{iik}\hat{\mathbf{e}}_k = 0 \qquad (39)$$

coordenadas cilíndricas

$$\begin{aligned}\nabla\times\mathbf{r} &= \frac{1}{\rho}\begin{vmatrix}\hat{\mathbf{e}}_\varrho & \rho\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z\\ \partial_\varrho & \partial_\theta & \partial_z\\ r_\varrho & \rho r_\theta & r_z\end{vmatrix} \quad \text{sendo o vetor posição} \quad \mathbf{r}=\varrho\hat{\mathbf{e}}_\varrho+z\hat{\mathbf{e}}_z\\ &= \hat{\mathbf{e}}_\varrho\left(\frac{1}{\rho}\partial_\theta r_z-\partial_z r_\theta\right)+\hat{\mathbf{e}}_\theta\left(\partial_z r_\varrho-\partial_\varrho r_z\right)+\frac{\hat{\mathbf{e}}_z}{\rho}\left(\partial_\varrho\rho r_\theta-\partial_\theta r_\varrho\right) \qquad (40)\\ &= \hat{\mathbf{e}}_\varrho\frac{1}{\rho}\partial_\theta z+\hat{\mathbf{e}}_\theta\partial_z\rho-\hat{\mathbf{e}}_\theta\partial_\varrho z-\hat{\mathbf{e}}_z\frac{1}{\rho}\partial_\theta\rho=0 \qquad (41)\end{aligned}$$

coordenadas esféricas

$$\begin{aligned}
\nabla \times \mathbf{r} &= \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin\theta\hat{\mathbf{e}}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ r_r & r\,r_\theta & r\sin\theta r_\phi \end{vmatrix} \quad \text{sendo o vetor posição} \quad \mathbf{r} = r\,\hat{\mathbf{e}}_r \\
&= \frac{\hat{\mathbf{e}}_r}{r\sin\theta}\Big(\partial_\theta \sin\theta r_\phi - \partial_\phi r_\theta\Big) + \frac{\hat{\mathbf{e}}_\theta}{r}\Big(\frac{1}{\sin\theta}\partial_\phi r_r - \partial_r r r_\phi\Big) + \frac{\hat{\mathbf{e}}_\phi}{r}\Big(\partial_r r r_\theta - \partial_\theta r_r\Big) \\
&= \frac{\hat{\mathbf{e}}_\theta}{r\sin\theta}\partial_\phi r - \hat{\mathbf{e}}_\phi \partial_\theta r = 0
\end{aligned}$$

....:: Solução (35) ::...

$$\nabla(\mathbf{1}\cdot\mathbf{r}) = \hat{\mathbf{e}}_i\partial_i(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k r_k) = \hat{\mathbf{e}}_i(\hat{\mathbf{e}}_j\cdot\hat{\mathbf{e}}_k)\partial_i r_k = \hat{\mathbf{e}}_i\delta_{jk}\partial_i r_k = \hat{\mathbf{e}}_i\partial_i r_j = \hat{\mathbf{e}}_i\delta_{ij} = \hat{\mathbf{e}}_j = \mathbf{1} \tag{42}$$

coordenadas cilíndrica

$$\begin{aligned}
\nabla(\mathbf{1}\cdot\mathbf{r}) &= \big(\hat{\mathbf{e}}_\varrho\partial_\varrho + \frac{1}{\varrho}\hat{\mathbf{e}}_\theta\partial_\theta + \hat{\mathbf{e}}_z\partial_z\big)\Big[\big(\hat{\mathbf{e}}_\varrho + \hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_z\big)\cdot\big(\varrho\hat{\mathbf{e}}_\varrho + z\hat{\mathbf{e}}_z\big)\Big] \\
&= \big(\hat{\mathbf{e}}_\varrho\partial_\varrho + \frac{1}{\varrho}\hat{\mathbf{e}}_\theta\partial_\theta + \hat{\mathbf{e}}_z\partial_z\big)\big(\varrho + z\big) \\
&= \hat{\mathbf{e}}_\varrho\partial_\varrho\varrho + \frac{1}{\varrho}\hat{\mathbf{e}}_\theta\partial_\theta\varrho + \hat{\mathbf{e}}_z\partial_z\varrho + \hat{\mathbf{e}}_\varrho\partial_\varrho z + \frac{1}{\varrho}\hat{\mathbf{e}}_\theta\partial_\theta z + \hat{\mathbf{e}}_z\partial_z z \\
&= \hat{\mathbf{e}}_\varrho + \hat{\mathbf{e}}_z
\end{aligned} \tag{43}$$

coordenadas esféricas

$$\begin{aligned}
\nabla(\mathbf{1}\cdot\mathbf{r}) &= \big(\hat{\mathbf{e}}_r\partial_r + \frac{1}{r}\hat{\mathbf{e}}_\theta\partial_\theta + \frac{1}{r\sin\theta}\hat{\mathbf{e}}_\phi\partial_\phi\big)\Big[\big(\hat{\mathbf{e}}_\varrho + \hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_z\big)\cdot\big(\varrho\hat{\mathbf{e}}_\varrho + z\hat{\mathbf{e}}_z\big)\Big] \\
&= \big(\hat{\mathbf{e}}_r\partial_r + \frac{1}{r}\hat{\mathbf{e}}_\theta\partial_\theta + \frac{1}{r\sin\theta}\hat{\mathbf{e}}_\phi\partial_\phi\big)\big(\varrho + z\big) \\
&= \hat{\mathbf{e}}_r\partial_r\varrho + \hat{\mathbf{e}}_r\partial_r z + \frac{1}{r}\hat{\mathbf{e}}_\theta\partial_\theta\varrho + \frac{1}{r}\hat{\mathbf{e}}_\theta\partial_\theta z + \frac{1}{r\sin\theta}\hat{\mathbf{e}}_\phi\partial_\phi\varrho + \frac{1}{r\sin\theta}\hat{\mathbf{e}}_\phi\partial_\phi z \\
&= 0
\end{aligned} \tag{44}$$

....:: Solução (36) ::...

$$(\mathbf{1}\cdot\nabla)\mathbf{r} = (\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j\partial_j)\hat{\mathbf{e}}_k r_k = (\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j)\hat{\mathbf{e}}_k\partial_j r_k = \delta_{ij}\hat{\mathbf{e}}_k\partial_j r_k = \hat{\mathbf{e}}_k\partial_i r_k = \hat{\mathbf{e}}_k\delta_{ik} = \hat{\mathbf{e}}_i = \mathbf{1} \tag{45}$$

coordenadas cilíndricas

$$\begin{aligned}
(\mathbf{1}\cdot\nabla)\mathbf{r} &= \Big[\big(\hat{\mathbf{e}}_\varrho + \hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_z\big)\cdot\big(\hat{\mathbf{e}}_\varrho\partial_\varrho + \frac{1}{\varrho}\hat{\mathbf{e}}_\theta\partial_\theta + \hat{\mathbf{e}}_z\partial_z\big)\Big]\big(\varrho\hat{\mathbf{e}}_\varrho + z\hat{\mathbf{e}}_z\big) \\
&= \big(\partial_\varrho + \frac{1}{\varrho}\partial_\theta + \partial_z\big)\big(\varrho\hat{\mathbf{e}}_\varrho + z\hat{\mathbf{e}}_z\big) \\
&= \partial_\varrho\varrho\hat{\mathbf{e}}_\varrho + \frac{1}{\varrho}\partial_\theta\varrho\hat{\mathbf{e}}_\varrho + \partial_z\varrho\hat{\mathbf{e}}_\varrho + \partial_\varrho z\hat{\mathbf{e}}_z + \frac{1}{\varrho}\partial_\theta z\hat{\mathbf{e}}_z + \partial_z z\hat{\mathbf{e}}_z \\
&= \hat{\mathbf{e}}_\varrho + \hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_z = \mathbf{1}
\end{aligned}$$

coordenadas esféricas

$$\begin{aligned}
(\mathbf{1}\cdot\nabla)\mathbf{r} &= \Big[\big(\hat{\mathbf{e}}_r\partial_r + \frac{1}{r}\hat{\mathbf{e}}_\theta\partial_\theta + \frac{1}{r\sin\theta}\hat{\mathbf{e}}_\phi\partial_\phi\big)\cdot\big(\hat{\mathbf{e}}_\varrho + \hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_\phi\big)\Big]\big(r\hat{\mathbf{e}}_r\big) \\
&= \big(\partial_r + \frac{1}{r}\partial_\theta + \frac{1}{r\sin\theta}\partial_\phi\big)\big(r\hat{\mathbf{e}}_r\big) \\
&= \partial_r r\hat{\mathbf{e}}_r + \frac{1}{r}\partial_\theta r\hat{\mathbf{e}}_r + \frac{1}{r\sin\theta}\partial_\phi r\hat{\mathbf{e}}_r \\
&= \hat{\mathbf{e}}_r + \hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_\phi = \mathbf{1}
\end{aligned}$$

6.0 - Sendo os vetores **a** e **b** constantes, calcule o divergente e o rotacional das equações abaixo

$$(\mathbf{a}\cdot\mathbf{r})\mathbf{b} \tag{46}$$
$$(\mathbf{a}\cdot\mathbf{r})\mathbf{r} \tag{47}$$
$$(\mathbf{a}\times\mathbf{r}) \tag{48}$$
$$\psi(r)(\mathbf{a}\times\mathbf{r}) \tag{49}$$
$$\mathbf{r}\times(\mathbf{a}\times\mathbf{r}) \tag{50}$$

....:: Solução (46) ::...

$$\begin{aligned}
\nabla\cdot(\mathbf{a}\cdot\mathbf{r})\mathbf{b} &= \hat{\mathbf{e}}_i\partial_i\cdot(\hat{\mathbf{e}}_j a_j\cdot\hat{\mathbf{e}}_k r_k)\hat{\mathbf{e}}_l b_l = \hat{\mathbf{e}}_i\partial_i\cdot(a_j r_k\delta_{jk})\hat{\mathbf{e}}_l b_l = a_k\partial_i r_k\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_l b_l \\
&= a_k\partial_i r_k\delta_{il}b_l = a_k b_l\partial_l r_k = a_k b_l\delta_{lk} = a_k b_k \\
&= (\mathbf{a}\cdot\mathbf{b})
\end{aligned} \tag{51}$$

$$\begin{aligned}
\nabla\times(\mathbf{a}\cdot\mathbf{r})\mathbf{b} &= \hat{\mathbf{e}}_i\partial_i\times(\hat{\mathbf{e}}_j a_j\cdot\hat{\mathbf{e}}_k r_k)\hat{\mathbf{e}}_l b_l = \hat{\mathbf{e}}_i\partial_i\times(a_j r_k\delta_{jk})\hat{\mathbf{e}}_l b_l = a_k\partial_i r_k\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_l b_l \\
&= a_k\partial_i r_k\epsilon_{ilm}\hat{\mathbf{e}}_m b_l = a_k b_l\partial_i r_k\epsilon_{ilm}\hat{\mathbf{e}}_m = a_k b_l\delta_{ik}\epsilon_{ilm}\hat{\mathbf{e}}_m = a_k b_l\epsilon_{klm}\hat{\mathbf{e}}_m \\
&= (\mathbf{a}\times\mathbf{b})
\end{aligned} \tag{52}$$

....:: Solução (47) ::...

$$\begin{aligned}
\nabla\cdot(\mathbf{a}\cdot\mathbf{r})\mathbf{r} &= \hat{\mathbf{e}}_i\partial_i\cdot(\hat{\mathbf{e}}_j a_j\cdot\hat{\mathbf{e}}_k r_k)\hat{\mathbf{e}}_l r_l = \hat{\mathbf{e}}_i\partial_i\cdot(a_j r_k\delta_{jk})\hat{\mathbf{e}}_l r_l = a_k\partial_i r_k\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_l r_l = a_k\partial_i r_k\delta_{il}r_l \\
&= a_k\partial_l r_k r_l = a_k r_k\partial_l r_l + a_k r_l\partial_l r_k = a_k r_k 3 + a_k r_l\delta_{lk} = 3a_k r_k + a_k r_k = 4a_k r_k \\
&= 4(\mathbf{a}\cdot\mathbf{r})
\end{aligned} \tag{53}$$

$$\begin{aligned}
\nabla\times(\mathbf{a}\cdot\mathbf{r})\mathbf{r} &= \hat{\mathbf{e}}_i\partial_i\times(\hat{\mathbf{e}}_j a_j\cdot\hat{\mathbf{e}}_k r_k)\hat{\mathbf{e}}_l r_l = \hat{\mathbf{e}}_i\partial_i\times(a_j r_k\delta_{jk})\hat{\mathbf{e}}_l r_l = a_k\partial_i r_k\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_l r_l \\
&= a_k\partial_i r_k\epsilon_{ilm}\hat{\mathbf{e}}_m r_l = a_k\partial_i r_k r_l\epsilon_{ilm}\hat{\mathbf{e}}_m = a_k r_k\partial_i r_l\epsilon_{ilm}\hat{\mathbf{e}}_m + a_k r_l\partial_i r_k\epsilon_{ilm}\hat{\mathbf{e}}_m \\
&= a_k r_k\delta_{il}\epsilon_{ilm}\hat{\mathbf{e}}_m + a_k r_l\delta_{ik}\epsilon_{ilm}\hat{\mathbf{e}}_m = a_k r_k\epsilon_{llm}\hat{\mathbf{e}}_m + a_k r_l\epsilon_{klm}\hat{\mathbf{e}}_m \\
&= (\mathbf{a}\times\mathbf{r})
\end{aligned} \tag{54}$$

....:: Solução (48) ::...

$$\begin{aligned}
\nabla\cdot(\mathbf{a}\times\mathbf{r}) &= \hat{\mathbf{e}}_i\partial_i\cdot(\hat{\mathbf{e}}_j a_j\times\hat{\mathbf{e}}_k r_k) = \hat{\mathbf{e}}_i\partial_i\cdot(a_j r_k\epsilon_{jkl}\hat{\mathbf{e}}_l) = a_j\partial_i r_k\epsilon_{jkl}\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_l \\
&= a_j\partial_i r_k\epsilon_{jkl}\delta_{il} = a_j\partial_l r_k\epsilon_{jkl} = a_j\delta_{lk}\epsilon_{jkl} = a_j\epsilon_{jkk} \\
&= 0
\end{aligned} \tag{55}$$

$$\nabla\times(\mathbf{a}\times\mathbf{r}) = \hat{\mathbf{e}}_i\partial_i\times(\hat{\mathbf{e}}_j a_j\times\hat{\mathbf{e}}_k r_k) = \hat{\mathbf{e}}_i\partial_i\times(a_j r_k\epsilon_{jkl}\hat{\mathbf{e}}_l) = a_j\partial_i r_k\epsilon_{jkl}\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_l \tag{56}$$
$$\begin{aligned}
&= a_j\partial_i r_k\epsilon_{jkl}\epsilon_{ilm}\hat{\mathbf{e}}_m = -a_j\partial_i r_k\epsilon_{jkl}\epsilon_{lim}\hat{\mathbf{e}}_m = -a_j\partial_i r_k(\delta_{ji}\delta_{km} - \delta_{jm}\delta_{ki})\hat{\mathbf{e}}_m \\
&= -a_j\partial_i r_k\delta_{ji}\delta_{km}e_m + a_j\partial_i r_k\delta_{jm}\delta_{ki}\hat{\mathbf{e}}_m = -a_i\partial_i r_m e_m + a_m\partial_i r_i\hat{\mathbf{e}}_m \\
&= -a_i\delta_{im}e_m + a_m 3\hat{\mathbf{e}}_m = -a_m e_m + 3a_m\hat{\mathbf{e}}_m = 2a_m\hat{\mathbf{e}}_m \\
&= 2\,\mathbf{a}
\end{aligned} \tag{57}$$

....:: Solução (49) ::...

$$\begin{aligned}
\nabla\cdot\psi(r)(\mathbf{a}\times\mathbf{r}) &= \hat{\mathbf{e}}_i\partial_i\cdot\psi(r)_i(\hat{\mathbf{e}}_j a_j\times\hat{\mathbf{e}}_k r_k) = \hat{\mathbf{e}}_i\partial_i\cdot\psi(r)_i(a_j r_k\epsilon_{jkm}\hat{\mathbf{e}}_m) \\
&= a_j\partial_i\psi(r)_i r_k\epsilon_{jkm}\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_m = a_j\partial_i\psi(r)_i r_k\epsilon_{jkm}\delta_{im} = a_j\partial_m\psi(r)_i r_k\epsilon_{jkm} \\
&= a_j\psi(r)_i\epsilon_{jkm}\partial_m r_k + a_j r_k\epsilon_{jkm}\partial_m\psi(r)_i = a_j\psi(r)_i\epsilon_{jkm}\delta_{mk} = a_j\psi(r)_i\epsilon_{jkk} \\
&= 0
\end{aligned} \tag{58}$$

$$
\begin{aligned}
\nabla \times \psi(r)(\mathbf{a}\times\mathbf{r}) &= \hat{\mathbf{e}}_i\partial_i \times \psi(r)_i(\hat{\mathbf{e}}_j a_j \times \hat{\mathbf{e}}_k r_k) = \hat{\mathbf{e}}_i\partial_i \times \psi(r)_i(a_j r_k \epsilon_{jkm}\hat{\mathbf{e}}_m)\\
&= a_j\partial_i\psi(r)_i r_k\epsilon_{jkm}\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_m = a_j\partial_i\psi(r)_i r_k\epsilon_{jkm}\epsilon_{iml}\hat{\mathbf{e}}_l\\
&= -a_j\partial_i\psi(r)_i r_k\epsilon_{jkm}\epsilon_{mil}\hat{\mathbf{e}}_l = -a_j\partial_i\psi(r)_i r_k(\delta_{ji}\delta_{kl}-\delta_{jl}\delta_{ki})\hat{\mathbf{e}}_l\\
&= -a_j\partial_i\psi(r)_i r_k\delta_{ji}\delta_{kl}\hat{\mathbf{e}}_l + a_j\partial_i\psi(r)_i r_k\delta_{jl}\delta_{ki}\hat{\mathbf{e}}_l\\
&= -a_i\partial_i\psi(r)_i r_l\hat{\mathbf{e}}_l + a_l\partial_i\psi(r)_i r_i\hat{\mathbf{e}}_l\\
&= -a_i\psi(r)_i\hat{\mathbf{e}}_l\partial_i r_l - a_i r_l\hat{\mathbf{e}}_l\partial_i\psi(r)_i + a_l\psi(r)_i\hat{\mathbf{e}}_l\partial_i r_i + a_l r_i\hat{\mathbf{e}}_l\partial_i\psi(r)_i\\
&= -a_i\psi(r)_i\hat{\mathbf{e}}_l\delta_{il} - r_l\hat{\mathbf{e}}_l a_i\partial_i\psi(r)_i + a_l\psi(r)_i\hat{\mathbf{e}}_l 3 + a_l\hat{\mathbf{e}}_l r_i\partial_i\psi(r)_i\\
&= -\psi(r)_i a_l\hat{\mathbf{e}}_l - r_l\hat{\mathbf{e}}_l a_i\partial_i\psi(r)_i + 3\psi(r)_i a_l\hat{\mathbf{e}}_l + a_l\hat{\mathbf{e}}_l r_i\partial_i\psi(r)_i\\
&= a_l\hat{\mathbf{e}}_l\big[2\psi(r)_i + r_i\partial_i\psi(r)_i\big] - r_l\hat{\mathbf{e}}_l a_i\partial_i\psi(r)_i\\
&= \mathbf{a}\big[2\psi(r) + \mathbf{r}\cdot\nabla\psi(r)\big] - \mathbf{r}\big[\mathbf{a}\cdot\nabla\psi(r)\big]
\end{aligned}
\tag{59}
$$

....:: Solução (50) ::..

$$
\begin{aligned}
\nabla\cdot\big[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})\big] &= \hat{\mathbf{e}}_i\partial_i\cdot\big[\hat{\mathbf{e}}_j r_j\times(\hat{\mathbf{e}}_k a_k\times\hat{\mathbf{e}}_l r_l)\big] = \hat{\mathbf{e}}_i\partial_i\cdot\big[\hat{\mathbf{e}}_j r_j\times(a_k r_l\epsilon_{klm}\hat{\mathbf{e}}_m)\big]\\
&= \hat{\mathbf{e}}_i\partial_i\cdot(r_j a_k r_l\epsilon_{klm}\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_m) = \hat{\mathbf{e}}_i\partial_i\cdot(r_j a_k r_l\epsilon_{klm}\epsilon_{jmn}\hat{\mathbf{e}}_n)\\
&= a_k\partial_i r_j r_l\epsilon_{klm}\epsilon_{jmn}\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_n = a_k\partial_i r_j r_l\epsilon_{klm}\epsilon_{jmn}\delta_{in}\\
&= a_k\partial_n r_j r_l\epsilon_{klm}\epsilon_{jmn} = -a_k\partial_n r_j r_l\epsilon_{klm}\epsilon_{mjn} = -a_k\partial_n r_j r_l(\delta_{kj}\delta_{ln}-\delta_{kn}\delta_{lj})\\
&= -a_k\partial_n r_j r_l\delta_{kj}\delta_{ln} + a_k\partial_n r_j r_l\delta_{kn}\delta_{lj} = -a_j\partial_n r_j r_n + a_n\partial_n r_j r_j\\
&= -a_j r_j\partial_n r_n - a_j r_n\partial_n r_j + 2\,a_n r_j\partial_n r_j = -3\,a_j r_j - a_j r_n\delta_{nj} + 2\,a_n r_j\delta_{nj}\\
&= -3\,a_j r_j - a_j r_j + 2\,a_j r_j = -2\,a_j r_j\\
&= -2(\mathbf{a}\cdot\mathbf{r})
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
\nabla\times\big[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})\big] &= \hat{\mathbf{e}}_i\partial_i\times\big[\hat{\mathbf{e}}_j r_j\times(\hat{\mathbf{e}}_k a_k\times\hat{\mathbf{e}}_l r_l)\big] = \hat{\mathbf{e}}_i\partial_i\times\big[\hat{\mathbf{e}}_j r_j\times(a_k r_l\epsilon_{klm}\hat{\mathbf{e}}_m)\big]\\
&= \hat{\mathbf{e}}_i\partial_i\times(r_j a_k r_l\epsilon_{klm}\hat{\mathbf{e}}_j\times\hat{\mathbf{e}}_m) = \hat{\mathbf{e}}_i\partial_i\times(r_j a_k r_l\epsilon_{klm}\epsilon_{jmn}\hat{\mathbf{e}}_n)\\
&= a_k\partial_i r_j r_l\epsilon_{klm}\epsilon_{jmn}\hat{\mathbf{e}}_i\times\hat{\mathbf{e}}_n = a_k\partial_i r_j r_l\epsilon_{klm}\epsilon_{jmn}\epsilon_{ino}\hat{\mathbf{e}}_o\\
&= -a_k\partial_i r_j r_l\epsilon_{klm}\epsilon_{jmn}\epsilon_{nio}\hat{\mathbf{e}}_o = -a_k\partial_i r_j r_l\epsilon_{klm}(\delta_{ji}\delta_{mo}-\delta_{jo}\delta_{mi})\hat{\mathbf{e}}_o\\
&= a_k\partial_i r_j r_l\epsilon_{klm}\delta_{jo}\delta_{mi}\hat{\mathbf{e}}_o - a_k\partial_i r_j r_l\epsilon_{klm}\delta_{ji}\delta_{mo}\hat{\mathbf{e}}_o\\
&= a_k\partial_m r_o r_l\epsilon_{klm}\hat{\mathbf{e}}_o - a_k\partial_i r_i r_l\epsilon_{klo}\hat{\mathbf{e}}_o\\
&= a_k r_o\epsilon_{klm}\hat{\mathbf{e}}_o\partial_m r_l + a_k r_l\epsilon_{klm}\hat{\mathbf{e}}_o\partial_m r_o - a_k r_i\epsilon_{klo}\hat{\mathbf{e}}_o\partial_i r_l - a_k r_l\epsilon_{klo}\hat{\mathbf{e}}_o\partial_i r_i\\
&= a_k r_o\epsilon_{klm}\hat{\mathbf{e}}_o\delta_{ml} + a_k r_l\epsilon_{klm}\hat{\mathbf{e}}_o\delta_{mo} - a_k r_i\epsilon_{klo}\hat{\mathbf{e}}_o\delta_{il} - a_k r_l\epsilon_{klo}\hat{\mathbf{e}}_o 3\\
&= a_k r_o\epsilon_{kll}\hat{\mathbf{e}}_o + a_k r_l\epsilon_{klo}\hat{\mathbf{e}}_o - a_k r_l\epsilon_{klo}\hat{\mathbf{e}}_o - 3a_k r_l\epsilon_{klo}\hat{\mathbf{e}}_o\\
&= -3a_k r_l\epsilon_{klo}\hat{\mathbf{e}}_o\\
&= -3(\mathbf{a}\times\mathbf{r}) = 3(\mathbf{r}\times\mathbf{a})
\end{aligned}
\tag{61}
$$
